Group Theory

Midterm Examination

September 8, 2026

Instructions: All questions carry ten marks.

1. Prove that a subgroup of a cyclic group is cyclic. Give an example of a group G whose every proper subgroup is cyclic but G is not cyclic.
2. Let S be a subset of a group G containing the identity element. Assume that distinct sets of the type aS, for $a \in G$ are disjoint. Prove that that S is a subgroup of G.
3. Define $Z(G)$, the center of a group G. Prove that the center of a non-abelian group can not have prime index.
4. Let G be a group and X be a non-empty set. Define a *group action* of G on X. Assume that the symmetric group group S_3 acts on two sets U and V both having cardinality 3 such that the action on U is transitive and V has two orbits. Let S_3 act on $U \times V$ by *diagonal action*, i.e. for any $g \in S_3$ and $(u, v) \in U \times V$, we have $g \cdot (u, v) = (g \cdot u, g \cdot v)$. Decompose $U \times V$ into orbits under this action.
5. Define the conjugacy action of a group G on itself. Classify all finite groups G having at most three orbits under the conjugacy action.
6. Let G be a finite group of order $p^n \cdot m$ such that p is prime and $(p, m) = 1$. Prove that G contains a subgroup of order p^n.